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The two-dimensional laminar jet in parallel streaming flow

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Solutions to the problem of a two-dimensional, laminar jet of incompressible fluid issuing into a uniform stream in the direction of the main flow are considered. Two co-ordinate-type expansions are developed. A direct expansion, when suitably transformed, predicts approximately the velocity along the plane of symmetry of the jet for all values of the abscissa, with a maximum error of 7·6 % far downstream from the origin. This error is established by comparison with a second, asymptotic expansion valid only at large values of the abscissa. The two expansions are subsequently joined, permitting an approximate determination of a constant which multiplies a third-order term in the asymptotic series and which initially remained unknown even after satisfying all boundary conditions imposed on these series.

The decay of velocity excess along the plane of symmetry of the jet is accelerated by the presence of the external stream.

1. Introduction

Flows caused by jets in an external streaming flow are of fundamental interest in applications ranging from aerodynamics to chemical processing. This type of flow is also of considerable theoretical interest. The problem had, nevertheless, not been solved mainly because of the fact that these are not similarity flows; therefore the equations of motion and continuity in their simplified, boundary-layer form cannot for this configuration be reduced to ordinary differential equations through the application of similarity transformations.

Let us consider a jet issuing from a slot in an infinite, two-dimensional expanse of parallel, uniformly streaming fluid. Let the velocity of the jet at the slot be larger in order of magnitude than that of the streaming flow anywhere else in the field (a 'strong jet'). It would then seem natural to analyse the resulting flow by an asymptotic treatment, starting with the known classical solution for a jet in otherwise quiescent fluid. The effect of the streaming flow would then be introduced as a small-perturbation effect.

At a considerable distance downstream of the slot, most of the jet momentum will have diffused into the expanse of the surrounding fluid. The velocity along the jet axis will, therefore, differ only by a small amount from the velocity of the streaming flow external to the jet ('weak jet'). This situation occurs close to the slot when the initial jet velocity is of the same order of magnitude as that of the external stream. Here one could try to analyse the flow asymptotically, starting from a solution which would be the equivalent in principle to Goldstein's (1933) classical treatment of the two-dimensional viscous wake.

Naturally, it is not *a priori* clear whether a satisfactory perturbation scheme can be developed for either of these cases. Indeed, as was first shown by Goldstein (1933), Stewartson (1957) and Chang (1961), the terms of such asymptotic expansions have to satisfy certain restrictions supplementary to the boundary conditions. For the analysis of the viscous, two-dimensional wake far downstream, Goldstein (1933) assumed an Oseen type of linearization in the boundary-layer equations for the fundamental term. Thereupon the first two terms of the regular perturbation expansion could be found explicitly. At the third term the procedure broke down. The solution for the perturbation of the velocity in the direction of streaming converged algebraically with distance normal to the plane of symmetry, although it can be argued upon physical grounds that convergence has to proceed exponentially for this case.† Goldstein suggested that the unsatisfactory result for the third term was a fault of the expansion assumed, and this point was later examined in much more detail by Stewartson (1957). It is well known that the asymptotic expansions proposed cannot be proven to be unique; Stewartson showed through an iterative solution to a form of the diffusion equation corresponding to Goldstein's problem that there is a contribution by a term of logarithmic order in the expansion parameter. This leads to a 'switchback' effect (Chang 1961): in order to be able to obtain higher approximations which accord with the principle of exponential decay of vorticity, terms of intermediate, logarithmic order *have* to appear in the expansion.

In the present analogous case of the weak jet, the presence of terms of logarithmic order in the expansion parameter will be assumed *a priori*. When all expansion terms are known analytically and explicitly, the form of the expansion can later be justified in detail. The 'weak jet' expansion, however, is not completely determined, since certain multiplicative constants which can be calculated neither through the application of boundary conditions nor integral theorems appear in the expansion. Indeed these constants may represent the influence of the upstream velocity (strong jet) on the flow far downstream, and can be determined only by appropriate joining of the two expansions. Because these expansions are of the co-ordinate type, one has to ensure that they have a common domain of convergence, or at least that one expansion has an infinite radius of convergence. This will be discussed in detail later. The method adopted for joining the two sequences is rather similar to that proposed by Van Dyke (1964*b*) for the case of a parabola in a uniform stream.

2. Analysis

The boundary-layer equations for the steady, incompressible jet in an infinite parallel stream in the absence of a pressure gradient are, in dimensionless form

$$\left.\begin{aligned}(U_\infty + u)\,\partial u/\partial x + v\,\partial u/\partial y &= \partial^2 u/\partial y^2,\\ \partial u/\partial x + \partial v/\partial y &= 0,\end{aligned}\right\} \tag{1}$$

† This fact has recently been formalized in *The Principle of Rapid Decay of Vorticity*, by Chang (1961), p. 834.

where

$$x = \frac{\tilde{x}}{\ell}, \quad y = \tilde{y}\left(\frac{U_{\text{ref}}}{\ell\nu}\right)^{\frac{1}{2}}, \quad u = \frac{\tilde{u}}{U_{\text{ref}}} - U_\infty, \quad v = \tilde{v}\left(\frac{\ell}{U_{\text{ref}}\,\nu}\right)^{\frac{1}{2}}, \quad U_\infty = \frac{\tilde{U}_\infty}{U_{\text{ref}}},$$

where $\tilde{x}$ and $\tilde{y}$ are Cartesian co-ordinates ($\tilde{x}$ in the direction of streaming), $\tilde{u}$ and $\tilde{v}$ the velocity components in the directions $\tilde{x}$ and $\tilde{y}$ respectively, ℓ and U_{ref} are an arbitrary reference length and velocity, ν is the kinematic viscosity and $\tilde{U}_\infty$ is the velocity of the undisturbed stream.

The boundary conditions to be fulfilled by (1) for this configuration are

$$\left.\begin{array}{llll} x > 0, & y = 0, & \partial u/\partial y = 0, & v = 0, \\ x \geqslant 0, & y = \infty, & u = 0. & \end{array}\right\} \tag{2}$$

The integrated form of (1) between the axis of symmetry and infinity is

$$\frac{d}{dx}\int_0^\infty u(u+U_\infty)\,dy = 0, \tag{3}$$

where the boundary conditions (2) were taken into account. The equivalent value of the integral may be obtained from a physical argument as

$$U_{\text{ref}}^2\,\ell\left(\frac{\nu}{U_{\text{ref}}\,\ell}\right)^{\frac{1}{2}}\int_0^\infty u(u+U_\infty)\,dy = \frac{1}{2\rho}(J - Q\tilde{U}_\infty) = \frac{J}{2\rho}\left(1-\frac{\tilde{U}_\infty}{\tilde{U}_J}\right), \tag{3a}$$

where J and Q are the jet momentum and mass flow per unit span, respectively, ρ is the fluid density, and $\tilde{U}_J$ is the average jet velocity at the slot.

Examination of (1) and (3) reveals that conditions for similarity exist only when $U_\infty = 0$ (i.e. a jet issuing into quiescent surrounding fluid). However, the flow tends asymptotically to become similar when $u \ll U_\infty$ (i.e. a small increment jet—'weak jet'). The two physically extreme situations yield two different similarity variables

$$\left.\begin{array}{lll} (1) & \xi \propto yx^{-\frac{2}{3}} & \text{when} \quad U_\infty = 0, \\ (2) & \eta \propto yx^{-\frac{1}{2}} & \text{when} \quad u \ll U_\infty. \end{array}\right\} \tag{4}$$

Equation (3) is satisfied when $u \propto x^{-\frac{1}{3}}$ in the first case and $u \propto x^{-\frac{1}{2}}$ in the second. When a jet of finite momentum, J, issues from a two-dimensional slot of width $t \to 0$, its velocity in the neighbourhood of the slot is considerably larger than that of the ambient stream. Consequently, it seems logical to perturb the similarity solution for the jet in still surroundings in order to accommodate a relatively slow parallel uniform stream. In terms of a similarity variable ξ, equation (3) will not be violated if a small perturbation parameter is defined as $\epsilon_s \propto U_\infty x^{\frac{1}{3}}$. Far downstream the jet momentum is diffused into the main flow and the velocity excess along the centre-line becomes small in comparison with the velocity of the external flow, irrespective of the initial strength of the jet. Thus the flow approaches the second similarity condition for which the velocity excess is regarded as a small perturbation (i.e. $\epsilon_w \propto x^{-\frac{1}{2}}$). Consequently, two separate expansions may be derived.

Comparison of the two perturbation parameters indicate that only co-ordinate-type expansions, for which the perturbation parameters are functions of x, may be obtained. The disparate powers of the co-ordinate associated with these small perturbation quantities make it impossible to replace the co-ordinate, x, by a single artificial parameter (Chang 1961), thus precluding any possibility of matching a finite number of terms of one expansion to a finite number of terms of the other. By extending the radius of convergence of each expansion so as to ensure satisfactory overlapping, certain characteristic quantities may be joined smoothly, provided some arbitrariness remains in one of the expansions. When a large number of terms of a particular expansion is calculated, all quantities of physical interest may be so joined. This process is essentially equivalent to an approximate matching of the two expansions, and provides an approximate solution for the entire flow field.

2.1. *The strong jet*

The asymptotic sequence representing the stream function, Ψ, when $U_\infty \ll 1$ is obtained by assuming a convenient similarity transform as follows:

$$\xi = \tfrac{1}{3}yx^{-\frac{2}{3}}, \quad \Psi(x,\xi) = 2x^{\frac{1}{3}} \sum_{k=0}^{\infty} \Delta_k(\epsilon_s) F_k(\xi), \tag{5}$$

i.e.

$$u = \frac{\partial \Psi}{\partial y} = \tfrac{2}{3}x^{-\frac{1}{3}} \sum_{k=0}^{\infty} \Delta_k(\epsilon_s) F'_k(\xi),$$

where the prime denotes differentiation with respect to ξ, and k is an integer. The coefficients Δ_k are functions of the small perturbation parameter ϵ_s and they cannot be determined *a priori*. ϵ_s may be defined as the ratio between the streaming velocity and the maximum velocity (at a given x) in the absence of an external stream. It will thus include the product $U_\infty x^{\frac{1}{3}}$ necessary to comply with (3) as discussed in the previous section:

$$\epsilon_s = \tfrac{3}{2} U_\infty x^{\frac{1}{3}}. \tag{6}$$

Introducing (5) into (1) and (3), we find

$$\Delta_k = \epsilon_s^k. \tag{6a}$$

In order to eliminate the possibility that functions which are transcendental in ϵ_s also occur, terms such as $\epsilon_s^3 \ln \epsilon_s$ were experimentally inserted. Predictably (Van Dyke 1964*a*), the equations for the associated functions F_k yielded zero identically as solutions. Inserting (6) and (6*a*) into (5) and the latter into (1), a set of ordinary differential equations is obtained by collecting terms of order $\Delta_0, \Delta_1, \ldots, \Delta_k$ for the functions $F_k(\xi)$ as follows:

$$\left.\begin{aligned} &F_0''' + 2F_0''F_0 + 2(F_0')^2 = 0,\\ &F_1''' + 2F_0F_1'' + 2F_0'F_1' + 4F_0''F_1 = -2(F_0' + 2\xi F_0''),\\ &F_2''' + 2F_0F_2'' + 6F_0''F_2 = -4(\xi + F_1)F_1'', \quad \text{etc.} \end{aligned}\right\} \tag{7}$$

One finds that the boundary conditions (2) reduce to

$$F_k(0) = F_k''(0) = F_k'(\infty) = 0. \tag{2a}$$

The first of equations (7) is easily solved by two quadratures to yield the classical solution for the jet in quiescent surrounding fluid (Schlichting 1933),

$$\left.\begin{aligned} F_0 &= \tanh\xi,\\ \text{and} \qquad u_0(x,y) &= \tfrac{2}{3}x^{-\frac{1}{3}}\operatorname{sech}^2\xi. \end{aligned}\right\} \tag{8}$$

Equation (3*a*) yields a convenient reference velocity

$$U_{\text{ref}} = \left\{\left[\frac{9}{16}\frac{J}{\rho}\left(1-\frac{Q\tilde{U}_\infty}{J}\right)\right]^2\frac{1}{\ell\nu}\right\}^{\frac{1}{3}}. \tag{9}$$

For the higher-order terms (3) shows that there has to be no contribution to the integral of orders from Δ_1 to Δ_k. For instance, for F_1' this implies

$$\int_0^\infty F_0'(1+2F_1')\,d\xi = 0, \tag{10}$$

and for F_2'

$$\int^\infty [2F_0'F_2'+F_1'(1+F_1')]\,d\xi = 0. \tag{10a}$$

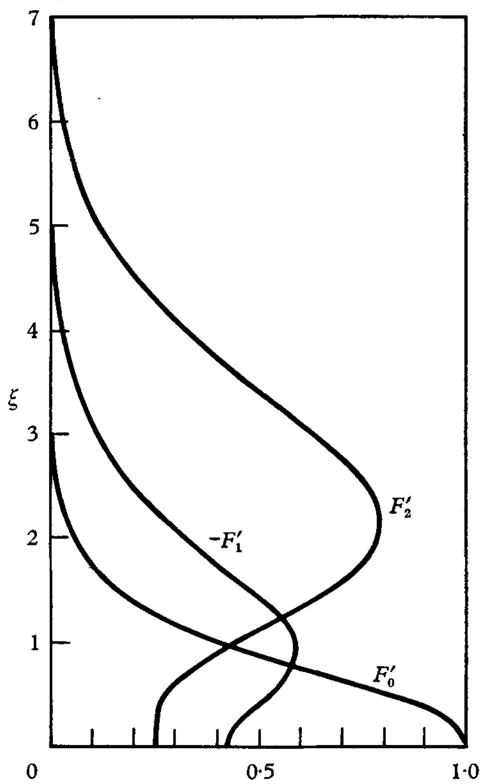

FIGURE 1. Functions F_0', F_1' and F_2' as given by (7).

It was not found possible to solve (7) in closed form. Solutions were therefore obtained numerically (figure 1) with (10) serving as a check of the numerical integration. F_1' and F_1, however, may be approximated by

$$\left.\begin{aligned} F_1'(\xi) &\approx (c^2-\eta^2)\operatorname{sech}^2\xi,\\ F_1(\xi) &\approx c^2\tanh\xi-\xi^2(1+\tanh\xi)+2\xi\ln\left(\tfrac{1}{2}\operatorname{sech}\xi\right)+\sum_{n=1}^{\infty}(-1)^n\frac{e^{-2n\xi}}{n^2}+\frac{\pi^2}{12},\end{aligned}\right\} \tag{11}$$

with $c^2 = -0{\cdot}4296$. See also Pozzi & Sabatini (1963).

Note that whereas the fundamental solution u_0, equation (8), represents a uniform approximation to the flow over the entire flow field, in that $\lim_{x\to\infty} u_0 \to U_\infty$ is satisfied, the same is no longer true for the higher-order approximations. In their present form the series diverge far downstream and the validity of the entire sequence is limited to the neighbourhood of the jet slot even for small value of U_∞. Thus an expansion such as (5) will constitute a very limited local solution which does not fulfil all the physical limitations of the problem. The use of terms of this solution is thus subject to severe limitations. Attempts, such as by Pozzi & Sabatini (1963) to graft a second-order term upon the basic solution may hold at best for very small x only (see figure 2).

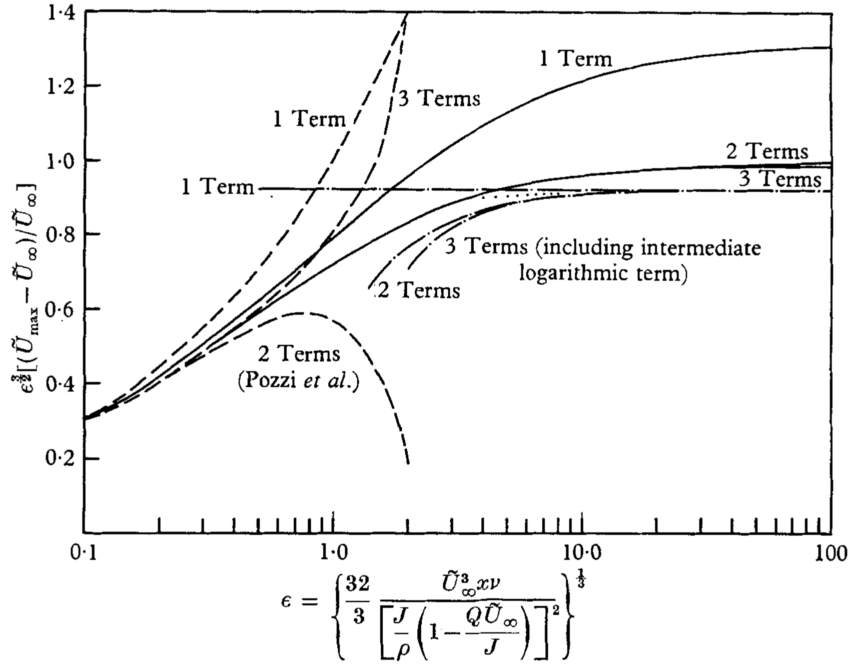

FIGURE 2. The attenuation of the velocity along the axis of symmetry. ....., composite solution; —·— asymptotic series for large x; – – – –, original series; ——, transformed series.

2.2 *The weak jet*

Far downstream U_∞ is considerably larger than $U_{\max}$ (the velocity excess along the axis of symmetry) and the jet becomes 'weak'. One may again obtain an approximate solution for the velocity in the form of an asymptotic expansion. There is, however, a fundamental difference in comparison with the case previously considered; the first term in the expansion will be of first order in the perturbation parameter. Therefore the differential equations for the various similarity functions will all be linear in this case, and the expansion will start with a term satisfying the linearized (Oseen) equation.

Proceeding as before, a small perturbation parameter is defined as

$$\epsilon_w = Ax^{-\frac{1}{2}}, \tag{12}$$

where A is a constant of proportionality and the reference velocity is chosen as $\tilde{U}_\infty$.† The similarity variable is given by

$$\eta = yx^{-\frac{1}{2}}. \tag{13}$$

† The choice of two disparate velocity scales yields a simple relationship between the similarity variables and the perturbation parameters of the strong and the weak jet (see (28) below).

An expression for the stream function ψ is assumed to have the form

$$\left.\begin{aligned}\psi(x,\eta) &= x^{\frac{1}{2}}\sum_{k=1}^{\infty}\delta_k(\epsilon_w)\,f_k(\eta),\\ \delta_1 &= \epsilon_w.\end{aligned}\right\} \tag{14}$$

The boundary conditions (2) remain essentially the same as for the strong jet

$$f_k(0) = f_k''(0) = f_k'(\infty) = 0. \tag{2b}$$

As already mentioned above the first approximation is obtained as the solution of a linear equation (Goldstein 1933). f_1 has to satisfy

$$\left.\begin{aligned}&f_1''' + \tfrac{1}{2}\eta f_1'' + \tfrac{1}{2}f_1' = 0,\\ \text{whence}\qquad &f_1' = e^{-\eta^2/4} \quad\text{and}\quad f_1 = \pi^{\frac{1}{2}}\operatorname{erf}(\tfrac{1}{2}\eta).\end{aligned}\right\} \tag{15}$$

The constant A in (12) may now be determined from the integral $(3a)$

$$A = \frac{1}{4\pi^{\frac{1}{2}}}\frac{J - Q\bar{U}_\infty}{\frac{1}{2}\rho\bar{U}_\infty^2\,\ell}\left(\frac{\bar{U}_\infty\,\ell}{\nu}\right)^{\frac{1}{2}}. \tag{16}$$

It will now be necessary to determine the form of the coefficients $\delta_k(\epsilon_w)$. These are not necessarily integer powers of ϵ_w as the only requirement is that

$$\lim_{\epsilon_w\to 0}\delta_{k+1}/\delta_k = 0.$$

The case of a viscous wake far downstream of a flat plate in an incompressible laminar flow is a problem which is similar to the one discussed here. For that problem Stewartson (1957) and Crane (1959) have shown the possibility for terms of logarithmic order in the expansion parameter to arise, usually in association with odd integer powers of ϵ_w. The logarithmic term and the algebraic term containing the same power of ϵ_w should be regarded as a single step in a successive approximation, since they are not independent. The purpose of the introduction of terms of logarithmic order in ϵ_w is to ensure that subsequent terms, algebraical in ϵ_w, will not yield solutions physically unacceptable. In the present problem the logarithmic term is necessary to eliminate a vorticity term decaying at only an algebraical rate with η. When a term logarithmic in ϵ_w appears for the first time in the expansion, its coefficient will satisfy a homogeneous differential equation. Therefore this function can only be determined up to an arbitrary constant. When the next term in the expansion is considered, its associated function will contain terms which decay algebraically with η. The constant left arbitrary in the previous term is now determined in a manner such that these algebraically decaying terms vanish identically. This process is referred to by Chang (1961) as 'switchback'. Thereafter, further logarithmic terms must appear in the series.

The expansion of $\psi(x,\eta)$ to order ϵ_w^3 is as follows:

$$\psi = x^{\frac{1}{2}}[\epsilon_w f_1 + (\epsilon_w)^2 f_2 + (\epsilon_w)^3\ln\epsilon_w f_3 + (\epsilon_w)^3 f_4 + \dots]. \tag{17}$$

The functions f_k are determined by the confluent hypergeometric equations,

$$\left.\begin{aligned} f_2''' + \tfrac{1}{2}\eta f_2'' + f_2' &= -\tfrac{1}{2}(f_1')^2, \\ f_3''' + \tfrac{1}{2}\eta f_3'' + \tfrac{3}{2} f_3' &= 0, \\ f_4''' + \tfrac{1}{2}\eta f_4'' + \tfrac{3}{2} f_4' &= \tfrac{1}{2}(f_2 f_1'' - 3f_1' f_2' - f_3'). \end{aligned}\right\} \tag{18}$$

The solutions satisfying the boundary conditions are

$$f_2' = -e^{-(\frac{1}{2}\eta)^2}[\tfrac{1}{2}e^{-(\frac{1}{2}\eta)^2} + \tfrac{1}{4}\eta\pi^{\frac{1}{2}}\operatorname{erf}(\tfrac{1}{2}\eta)], \tag{19}$$

$$f_2 = \tfrac{1}{2}e^{-(\frac{1}{2}\eta)^2}\pi^{\frac{1}{2}}\operatorname{erf}(\tfrac{1}{2}\eta) - (\tfrac{1}{2}\pi)^{\frac{1}{2}}\operatorname{erf}(\eta/\sqrt{2}), \tag{19a}$$

$$f_3' = \alpha(1-\tfrac{1}{2}\eta^2)e^{-(\frac{1}{2}\eta)^2}, \tag{20}$$

$$f_3 = \alpha\eta e^{-(\frac{1}{2}\eta)^2}, \tag{20a}$$

$$\begin{aligned} f_4' = {} & -\tfrac{1}{2}\alpha\eta\pi^{\frac{1}{2}}\operatorname{erf}(\tfrac{1}{2}\eta) - \tfrac{1}{8}\eta(\tfrac{1}{3}\pi)^{\frac{1}{2}}\operatorname{erf}(\tfrac{1}{2}\sqrt{3}\,\eta) + \left\{\left[-\frac{1}{16}\sqrt{\frac{\pi}{3}}\int_0^\eta \operatorname{erf}(\tfrac{1}{2}\sqrt{3}\mu)\,e^{(\frac{1}{2}\mu)^2}d\mu \right.\right. \\ & \left. - \tfrac{1}{4}\alpha\sqrt{\pi}\int_0^\eta \operatorname{erf}(\tfrac{1}{2}\mu)\,e^{(\frac{1}{2}\mu)^2}d\mu - (\pi/32)\operatorname{erf}^2(\tfrac{1}{2}\eta) + \beta\right](2-\eta^2) \\ & \left. + [\tfrac{1}{4}e^{-(\frac{1}{2}\eta)^2}\sqrt{\pi}\operatorname{erf}(\tfrac{1}{2}\eta) + (\tfrac{1}{2}\pi)^{\frac{1}{2}}\operatorname{erf}(\eta/\sqrt{2})]\,\tfrac{1}{2}\eta + \alpha + \tfrac{1}{2}\exp[-(\eta/\sqrt{2})^2]\right\} e^{-(\frac{1}{2}\eta)^2}, \end{aligned} \tag{21}$$

where the coefficients α and β cannot be determined from any conservation law, such as the momentum integral ($3a$). In order to ensure exponential decay of f_4' we must have $\alpha = -1/4\sqrt{3}$. However, β remains arbitrary.

$$\begin{aligned} f_4 = {} & -(1/4\sqrt{3})\pi^{\frac{1}{2}}\operatorname{erf}(\tfrac{1}{2}\eta) + \tfrac{7}{4}(\tfrac{1}{3}\pi)^{\frac{1}{2}}\operatorname{erf}(\tfrac{1}{2}\sqrt{3}\,\eta) - e^{-(\frac{1}{2}\eta)^2}\left\{(\tfrac{1}{2}\pi)^{\frac{1}{2}}\operatorname{erf}(\eta/\sqrt{2})\right. \\ & + \tfrac{1}{4}e^{-(\frac{1}{2}\eta)^2}\pi^{\frac{1}{2}}\operatorname{erf}(\tfrac{1}{2}\eta) + (\pi/16)\,\eta\operatorname{erf}^2(\tfrac{1}{2}\eta) - 2\beta\eta \\ & \left. + \tfrac{1}{8}\eta(\tfrac{1}{3}\pi)^{\frac{1}{2}}\int_0^\eta e^{(\frac{1}{2}\mu)^2}[\operatorname{erf}(\tfrac{1}{2}\sqrt{3}\,\mu) - \operatorname{erf}(\tfrac{1}{2}\mu)]\,d\mu\right\}. \end{aligned} \tag{21a}$$

We are now in a position to write out the velocity components u and v to order $(\epsilon_w)^4$:

$$\begin{aligned} u = {} & \epsilon_w e^{-(\frac{1}{2}\eta)^2}\left\{1 - \epsilon_w[\tfrac{1}{2}e^{-(\frac{1}{2}\eta)^2} + \tfrac{1}{4}\eta\pi^{\frac{1}{2}}\operatorname{erf}(\tfrac{1}{2}\eta)] - (1/4\sqrt{3})(\epsilon_w)^2\ln\epsilon_w(1-\tfrac{1}{2}\eta^2)\right. \\ & + (\epsilon_w)^2\left\{\left[\tfrac{1}{16}(\tfrac{1}{3}\pi)^{\frac{1}{2}}\int_0^\eta (\operatorname{erf}(\tfrac{1}{2}\mu) - \operatorname{erf}(\tfrac{1}{2}\sqrt{3}\,\mu))\,e^{(\frac{1}{2}\mu)^2}d\mu - (\pi/32)\operatorname{erf}^2(\tfrac{1}{2}\eta) + \beta\right].(2-\eta^2)\right. \\ & \left.\left. + [\tfrac{1}{4}e^{-(\frac{1}{2}\eta)^2}\sqrt{\pi}\operatorname{erf}(\tfrac{1}{2}\eta) + (\tfrac{1}{2}\pi)^{\frac{1}{2}}\operatorname{erf}(\eta/\sqrt{2})]\,\tfrac{1}{2}\eta - (1/4\sqrt{3}) + \tfrac{1}{2}\exp[-(\eta/\sqrt{2})^2]\right\}\right\} \\ & + (\epsilon_w)^3\,\tfrac{1}{8}\eta(\tfrac{1}{3}\pi)^{\frac{1}{2}}[\operatorname{erf}(\tfrac{1}{2}\eta) - \operatorname{erf}(\tfrac{1}{2}\sqrt{3}\,\eta)] + \dots, \end{aligned} \tag{22}$$

$$v = \tfrac{1}{2}x^{-\frac{1}{2}}\eta\epsilon_w e^{-(\frac{1}{2}\eta)^2}\{1 - \epsilon_w[\tfrac{1}{2}e^{-(\frac{1}{2}\eta)^2} + \tfrac{1}{4}\eta\pi^{\frac{1}{2}}\operatorname{erf}(\tfrac{1}{2}\eta)]\} + \dots, \tag{23}$$

since $x^{-\frac{1}{2}} \propto \epsilon_w$ and the logarithmic term is considered as part of the higher-order term.

To the order considered v is fully determined since the arbitrary constant β appears only in a higher-order term. (23) shows that v is antisymmetrical in η, and remains finite when $\eta \to \infty$. The error is of order $(\epsilon_w)^3$. This is a common shortcoming arising from the application of boundary-layer approximations to the problem under consideration. In principle, one may obtain higher-order approximations for u and v which will contain more arbitrary constants each time the inclusion of a new homogeneous equation becomes necessary.

2.3. *Transformation of the strong jet expansion*

Having computed the functions F'_k (figure 1), the velocity along the axis of symmetry may be obtained from (5) as

$$\left.\begin{aligned} U_{\max} &= \tfrac{2}{3}x^{-\frac{1}{3}}(1-0{\cdot}4296\epsilon_s+0{\cdot}2463\epsilon_s^2-\ldots),\\ \text{or}\qquad \epsilon_s^{\frac{3}{2}}\frac{\bar{U}_{\max}-\bar{U}_\infty}{\bar{U}_\infty} &= \epsilon_s^{\frac{1}{2}}(1-0{\cdot}4296\epsilon_s+0{\cdot}2463\epsilon_s^2-\ldots).\end{aligned}\right\} \tag{24}$$

It is quite obvious (figure 2) that (24) is only useful when $\epsilon_s \ll 1$.

As is often the case with expansions of this type, the restriction on convergence of these series may have arisen from the choice of the expansion quantity and the co-ordinate system rather than from the function which they represent (e.g. Meksyn 1961). Introduction of a non-linear transformation suggested by Shanks (1955) results in a partial sum given by the first three terms of the series, (24),

$$e_1 = \frac{0{\cdot}4296+(0{\cdot}2463-0{\cdot}4296^2)\,\epsilon_s}{0{\cdot}4296+0{\cdot}2463\epsilon_s}. \tag{25}$$

The denominator of this fraction vanishes at $\epsilon_s = -1{\cdot}74$, which suggests the existence of a singularity in that vicinity. This leads to a choice of a new perturbation parameter

$$\bar{\epsilon}_s = \frac{\epsilon_s}{1{\cdot}7+\epsilon_s}, \tag{26}$$

in terms of which the original series may be rearranged to give

$$\epsilon_s^{\frac{3}{2}}(\bar{U}_{\max}-\bar{U}_\infty)/\bar{U}_\infty = 1{\cdot}3038\bar{\epsilon}_s^{\frac{1}{2}}(1-0{\cdot}2303\bar{\epsilon}_s-0{\cdot}0086\bar{\epsilon}_s^2-\ldots). \tag{27}$$

Following Van Dyke (1964*b*), it is suggested that (27) represents the correct velocity everywhere along the centre-line of the jet, and that the series converge to the asymptotic value calculated by the first term of the weak jet expansion. To test this conjecture it is necessary to assume that the origins of the co-ordinate systems in both expansions are identical. This assumption is compatible with the original statement that the jet of finite momentum per unit span originates from a slot of width $t \to 0$. Consequently the relationship between the respective perturbation parameters is obtained from (6), (9), (12) and (16);

$$\epsilon_w = (8/3\pi)^{\frac{1}{2}}\epsilon_s^{-\frac{3}{2}} = 0{\cdot}9214\epsilon_s^{-\frac{3}{2}}. \tag{28}$$

The first term in the weak-jet solution gives

$$\left.\begin{aligned} (\bar{U}_{\max}-\bar{U}_\infty)/\bar{U}_\infty &= \epsilon_w,\\ \text{or}\qquad \epsilon_s^{\frac{3}{2}}(\bar{U}_{\max}-\bar{U}_\infty)/\bar{U}_\infty &= 0{\cdot}9214.\end{aligned}\right\} \tag{29}$$

The successive partial sums of the first three terms of (27) when $\bar{\epsilon}_s \to 1$ (i.e. $x \to \infty$) give

$$1{\cdot}3038, \quad 1{\cdot}0037, \quad 0{\cdot}992. \tag{30}$$

It seems quite likely that this series converges eventually to the required value of 0·9214.

The transformed series is plotted in figure 2 for comparison.

2.4. *Joining the direct and inverse expansion and construction of a composite solution*

It is now possible to evaluate approximately the unknown constant β appearing in (21), by joining the two expansions along the centre-line of the jet. Letting $y = 0$ in (22) yields

$$\left.\begin{aligned} U_{\max} &= \epsilon_w - \tfrac{1}{2}\epsilon_w^2 - \frac{1}{4\sqrt{3}}\,\epsilon_w^3 \ln \epsilon_w + \left(2\beta - \frac{1}{4\sqrt{3}} + \tfrac{1}{2}\right)\epsilon_w^3 + \ldots, \\ &\text{or} \\ \epsilon_s^{\frac{3}{2}}(\bar{U}_{\max} - \bar{U}_\infty)/\bar{U}_\infty &= (8/3\pi)^{\frac{1}{2}}\left[1 - \tfrac{1}{2}\epsilon_w - \frac{1}{4\sqrt{3}}\,\epsilon_w^2 \ln \epsilon_w + \left(2\beta - \frac{1}{4\sqrt{3}} + \tfrac{1}{2}\right)\epsilon_w^2 + \ldots\right]. \end{aligned}\right\} \quad (31)$$

From (26) and (28) the following relationship is obtained:

$$\epsilon_w = \left(\frac{8}{3\pi}\right)^{\frac{1}{2}}\left(\frac{1-\bar{\epsilon}_s}{1\cdot 7\bar{\epsilon}_s}\right)^{\frac{3}{2}}. \quad (32)$$

Substituting (32) into (31) and letting $\bar{\epsilon}_s \to 1$ gives

$$\begin{aligned} \epsilon_s^{\frac{3}{2}}(\bar{U}_{\max} - \bar{U}_\infty)/\bar{U}_\infty + 0\cdot 0345(1-\bar{\epsilon}_s)^3 \ln(1-\bar{\epsilon}_s) &= 0\cdot 9214 - 0\cdot 1961(1-\bar{\epsilon}_s)^{\frac{3}{2}} \\ &\quad + (0\cdot 3205\beta + 0\cdot 0769)(1-\bar{\epsilon}_s)^3. \end{aligned} \quad (33)$$

The logarithmic term in (33) may be expanded as follows:

$$(1-\bar{\epsilon}_s)^3 \ln(1-\bar{\epsilon}_s) = (1 - 3\bar{\epsilon}_s + 3\bar{\epsilon}_s^2 - \bar{\epsilon}_s^3)(-\bar{\epsilon}_s - \tfrac{1}{2}\bar{\epsilon}_s^2 - \tfrac{1}{3}\bar{\epsilon}_s^3 - \ldots) \quad \text{for} \quad 0 \leqslant \bar{\epsilon}_s \leqslant 1.$$

The direct expansion (27) is substituted for the remaining term on the left-hand side of (33):

$$\begin{aligned} \epsilon_s^{\frac{3}{2}}(\bar{U}_{\max} - \bar{U}_\infty)/\bar{U}_\infty + 0\cdot 0345(1-\bar{\epsilon}_s)^3 \ln(1-\bar{\epsilon}_s) &= 1\cdot 3038\bar{\epsilon}_s^{\frac{1}{2}} - 0\cdot 0345\bar{\epsilon}_s \\ &\quad - 0\cdot 3003\bar{\epsilon}_s^{\frac{3}{2}} + 0\cdot 0862\bar{\epsilon}_s^2 - 0\cdot 0112\bar{\epsilon}_s^{\frac{5}{2}}. \end{aligned} \quad (34)$$

Equation (34) is now expanded formally for small $(1-\bar{\epsilon}_s)$ and compared with (33) to obtain the unknown constant

$$0\cdot 3205\beta + 0\cdot 0769 = -0\cdot 0815 - 0\cdot 0188 + 0\cdot 0035 - \ldots, \quad \text{or} \quad \beta \approx -0\cdot 54. \quad (35)$$

Having obtained the numerical value of β the function f_4' becomes fully determined. For convenience the functions f_1' through f_4' are plotted in figure 3.

Since the partial sum of the first three terms of the direct expansion deviates by 7·6 % from the asymptotic series, when $x \to \infty$, it is useful to construct a composite solution which will depart from the direct series at some finite x and converge to the correct asymptotic value. No fundamental significance beyond this fact need be attached to this construction. Recall that the velocity along the plane of symmetry as given by the asymptotic solution for large x is

$$\epsilon_s^{\frac{3}{2}}\frac{\bar{U}_{\max} - \bar{U}_\infty}{\bar{U}_\infty} = 0\cdot 9214\left(1 - \tfrac{1}{2}\epsilon_w - \frac{1}{4\sqrt{3}}\,\epsilon_w^2 \ln \epsilon_w - 0\cdot 7281\epsilon_w^2 \ldots\right), \quad (36)$$

while the direct co-ordinate expansion gives

$$\epsilon_s^{\frac{3}{2}} \frac{\tilde{U}_{\max} - \tilde{U}_\infty}{\tilde{U}_\infty} = 1{\cdot}3038\bar{\epsilon}_s^{\frac{1}{2}}(1 - 0{\cdot}2303\bar{\epsilon}_s - 0{\cdot}0086\bar{\epsilon}_s^2 \ldots) \tag{37}$$

and when re-expanded for small ϵ_w it becomes

$$\epsilon_s^{\frac{3}{2}}(\tilde{U}_{\max} - \tilde{U}_\infty)/\tilde{U}_\infty = 0{\cdot}992 - 0{\cdot}313\epsilon_w^{\frac{2}{3}} - 0{\cdot}396\epsilon_w^{\frac{4}{3}} - 1{\cdot}824\epsilon_w^2 \ldots \tag{38}$$

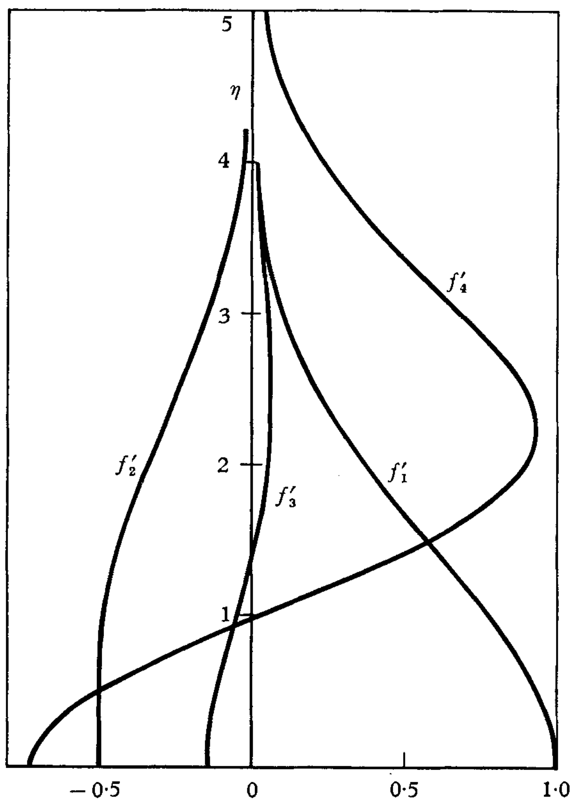

FIGURE 3. Functions f_1' to f_4' as given by (15), (19), (20) and (21).

The composite expansion is obtained by adding (36) to (37) and subtracting (38). The resulting $\epsilon_s^{\frac{3}{2}}(\tilde{U}_{\max} - \tilde{U}_\infty)/\tilde{U}_\infty$ is plotted in figure 2. As seen from the figure the latter is probably more accurate, when $\epsilon_s > 3{\cdot}5$, than each of its original constituents. A composite expansion of this sort is not limited to the velocity along the jet axis only, but can be extended to the entire velocity profile to provide a smooth transition from the strong jet to the weak jet expansion.

3. Discussion

An approximate solution for the laminar jet in a parallel uniform stream was obtained. The first three terms of the direct co-ordinate expansion, when properly transformed, overestimate the velocity along the plane of symmetry by only 7·6 %. This result compares quite favourably with the expansion for the parabola in a uniform stream (Van Dyke 1964*b*), where the partial sum of the first six terms agreed with the asymptotic value to within 15%. Probably, when more terms are calculated and properly transformed, the above-mentioned error could be reduced. This process may require repeated transformations (Shanks 1955) and seems hardly justified. The apparent existence of a singularity at $\epsilon_s \approx -1{\cdot}7$ in the original series may indicate that the variables have not been chosen in the most natural way (Van Dyke 1964*a*). However, a more natural co-ordinate system is not indicated from the geometry. (This should not be confused with optimal co-ordinates (Kaplun 1954).)

Joining two co-ordinate-type expansions is not as exact a process as matching two parameter-type asymptotic expansions. Consequently, the value of β as determined in § 2.4 is only approximate.

The applicability of the boundary-layer approximations has to be examined so as to ensure that terms of the order of the higher perturbations have not been neglected at the very start, when introducing these approximations into the momentum equations. One method of investigating qualitatively the range of applicability of the boundary-layer approximations is to delimit that part of the flow field over which the ratio v/u is small. Another way is to estimate the relative magnitude of the term $\partial^2 u/\partial x^2$ which is neglected on the right-hand side of (1). Using either of these estimates it is found that the ratios become of order unity at very large ξ, η respectively, and also for very small values of the Reynolds modulus based upon U_{ref} and $\tilde{x}$. In § 2.2 we have already drawn attention to the fact that a finite velocity v at $\eta \to \infty$ is predicted.

For the range in which strong jet behaviour is dominant, ϵ_s increases as the ratio v/u decreases ($v/u \propto x^{-\frac{1}{3}}$). Therefore, here the boundary-layer approximations will apply to the higher-order solution to the same accuracy that they apply to the fundamental solution of the strong jet in still air. For the region where weak jet behaviour predominates the situation is more involved, confining the validity of higher-order approximations to the region of small η. Fortunately, the region of practical interest coincides with the region in which the boundary-layer approximations apply.†

Far away from the blowing slot, where the flow is governed by the weak jet solution, the present analysis could be extended to the external flow field (i.e. large η) through the application of matched asymptotic expansions in a manner shown by Chang (1961). From a practical point of view one is not predominantly interested in regions far downstream. This is so, because relatively soon downstream the jet will become turbulent, and the basic premises of the theory will cease to be valid. Therefore, it would seem unnecessary to consider within the context of the present work the outer region as well. The distance at which transition to turbulent flow occurs will not be a function of a single Reynolds number based on $\tilde{U}_\infty$, say, since the ratio $\tilde{U}_\infty/\tilde{U}_J$ will have some influence. For the strong jet in still fluid a limited amount of experimental information is available which enables the estimation of the distance from the orifice where the change-over is likely to take place (cf. Da Costa Andrade 1939). For the jet in an external streaming flow no such information is apparently available.

† The author is particularly indebted to Dr Z. Rotem for this comment.

The form of the expansions stipulated in the paper appears to be entirely self-consistent to the order of approximation calculated. Terms of intermediate order $\epsilon_w^n \ln \epsilon_w (n > 3)$ will now have to appear regularly further on. When either these or the purely algebraical terms fail to accord with the principle of rapid decay of vorticity, a further term of the form $\epsilon_w^n \ln^m \epsilon_w \, (m > 1)$ will have to be introduced. The latter will contain new constants, in principle determinable only from the 'strong jet' solution.

In conclusion, the method of solution outlined in the paper appears to be rather suitable to the investigation of jets in streaming motion. The same method has already been applied to axisymmetrical jets, and with suitable supplementary assumptions it may also yield information on the behaviour of turbulent jets in streaming flow.

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